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Letter to the Editor

Comparison of higher order versions of the method of multiple scales for an odd non-linearity problem

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1. Introduction

The method of multiple scales is one of the important perturbation techniques widely used. The method yields transient solutions as well as steady state solutions in contrast to some other techniques which yield only the steady state solution.

When employing higher order expansions, slightly different versions of the method appear in the literature. The oldest version, called reconstitution method, is due to Nayfeh (see, for example, Refs. [1–3]). Generally speaking, in this method, for primary resonances, the damping and forcing terms are re-ordered such that they balance the effect of non-linearities. The nearness of the external excitation frequency to one of the natural frequencies is represented by using only one correction term. The time derivatives for each time scale do not vanish separately, but their sum vanishes for finding the steady state solutions. In contrast, Rahman and Burton [4] showed that the reconstitution method (which will be called MMS I) cannot capture well the steady state Lindstedt–Poincare solutions. MMS I yielded extra solutions which are not physical for the simple duffing oscillator. Rahman and Burton [4] then suggested an alternative version (MMS II) to handle the problem. The excitation frequency and the damping should be expanded in a series and require that each time-scale derivative vanish independently. This method was presented for finding the steady state solutions. However, the unsteady solutions cannot be retrieved using the method. Boyacỳ and Pakdemirli [5] applied this new version as well as MMS I to partial differential equations with arbitrary quadratic and cubic non-linearities and found similar results to Ref. [4]. Hassan [6] applied MMS II to the case of superharmonic resonances and compared his results with the harmonic balance method. Later, Lee and Lee [7] improved MMS II by showing how to calculate unsteady solutions as well as the steady state solutions (MMS II modified). Similar to MMS I, the suggested modified version make series expansions unnecessary for the frequency, damping and excitation amplitude. In this version, damping and excitation are scaled to appear in the first non-linear order. In MMS II, only the steady state solutions can be retrieved

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whereas in this modified version transient solutions can also be obtained. In establishing this, time derivatives are taken to be non-zero only on their corresponding level of approximation, i.e., D_1 terms are non-zero on the first level of approximation but vanish on the second level of approximation.

In treating partial differential equations using MMS, besides selecting a version, one has to select one of the two choices: (1) discretize the equations first and then apply MMS (discretization-perturbation method) or (2) apply MMS directly to the partial differential system (direct-perturbation method). The advantages of selecting the latter choice has been widely discussed [2,5, 8–13].

In this work, an odd non-linearity problem will be treated using MMS I and MMS II modified. The model considered is

$$\ddot{x} + \mu\dot{x} + x + \alpha x^3 + \beta x^5 = F\cos\Omega t, \tag{1}$$

where x is the amplitude of vibrations, μ is the viscous damping, α and β are the non-linearity coefficients, and F and Ω are the external excitation amplitude and frequency, respectively. Dot denotes differentiation with respect to time variable t. Approximate solutions will be derived using two higher order versions and comparisons between the methods as well as with direct numerical calculations will be presented.

2. Approximate solution by MMS I

The non-linearities are reordered such that their effects appear at different orders

$$\alpha = \varepsilon\bar{\alpha}, \quad \beta = \varepsilon^2\bar{\beta}. \tag{2}$$

In this version, the excitation amplitude and damping are ordered such that they balance the non-linearity at the last order. Primary resonances are assumed for the external excitation frequency

$$F = \varepsilon^2 f, \quad \mu = \varepsilon^2\bar{\mu}, \quad \Omega = 1 + \varepsilon^2\sigma. \tag{3}$$

This choice of ordering is the best possible choice as will be explained at the end of this section and Section 4. The approximate expansion is

$$x(t;\varepsilon) = x_0(T_0, T_1, T_2) + \varepsilon x_1(T_0, T_1, T_2) + \varepsilon^2 x_2(T_0, T_1, T_2) + \cdots, \tag{4}$$

where $T_n = \varepsilon^n t$. Time derivatives in terms of fast and slow time scales are

$$\begin{aligned} \mathrm{d}/\mathrm{d}t &= D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots, \\ \mathrm{d}^2/\mathrm{d}t^2 &= D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2(D_1^2 + 2D_0 D_2) + \cdots, \end{aligned} \tag{5}$$

where $D_n = \partial/\partial T_n$. Substituting all into the original equation and separating terms at each order yields

$$D_0^2 x_0 + x_0 = 0, \tag{6}$$

$$D_0^2 x_1 + x_1 = -2D_0 D_1 x_0 - \bar{\alpha} x_0^3, \tag{7}$$

$$D_0^2 x_2 + x_2 = -2D_0 D_1 x_1 - (D_1^2 + 2D_0 D_2) x_0 - \bar{\mu} D_0 x_0 - 3\bar{\alpha} x_0^2 x_1 - \bar{\beta} x_0^5 + f \cos \Omega T_0. \tag{8}$$

The first order solution is

$$x_0 = A(T_1, T_2) \exp(\mathrm{i}T_0) + \text{c.c.}, \tag{9}$$

where c.c. stands for complex conjugates of preceding terms. Removing the secular terms at the next order requires

$$2\mathrm{i}D_1 A = -3\bar{\alpha} A^2 \bar{A}. \tag{10}$$

The solution at this order is

$$x_1 = \frac{\bar{\alpha}}{8}(A^3 \exp(3\mathrm{i}T_0) + \text{c.c.}). \tag{11}$$

At the last order, the secular terms are removed if

$$2\mathrm{i}D_2 A = -\bar{\mu}\mathrm{i}A - \left(10\bar{\beta} - \tfrac{15}{8}\bar{\alpha}^2\right) A^3 \bar{A}^2 + \frac{f}{2} \exp(\mathrm{i}\sigma T_2). \tag{12}$$

The evolution of complex amplitudes in real time would then be

$$2\mathrm{i}\frac{\mathrm{d}A}{\mathrm{d}t} = 2\mathrm{i}(\varepsilon D_1 A + \varepsilon^2 D_2 A) + \cdots \tag{13}$$

or

$$2\mathrm{i}\frac{\mathrm{d}A}{\mathrm{d}t} = -\mu\mathrm{i}A - 3\alpha A^2 \bar{A} - \left(10\beta - \tfrac{15}{8}\alpha^2\right) A^3 \bar{A}^2 + \frac{F}{2} \exp(\mathrm{i}(\Omega - 1)t). \tag{14}$$

Note that all system parameters are expressed in their original form since the original equation does not depend on the artificially introduced perturbation parameter. Using the polar form

$$A = \tfrac{1}{2} a \exp(\mathrm{i}\lambda), \tag{15}$$

the approximate solution and amplitude phase modulation equations can be written in the form

$$x = a \cos(\Omega t - \gamma) + \frac{\alpha a^3}{32} \cos(3\Omega t - 3\gamma) + \cdots, \tag{16}$$

$$\dot{a} = -\frac{\mu}{2} a + \frac{F}{2} \sin \gamma, \tag{17}$$

$$a\dot{\gamma} = a(\Omega - 1) - \tfrac{3}{8}\alpha a^3 - \frac{1}{32}\left(10\beta - \tfrac{15}{8}\alpha^2\right) a^5 + \frac{F}{2} \cos \gamma, \tag{18}$$

where

$$\gamma = (\Omega - 1)t - \lambda. \tag{19}$$

In the case of steady state motion, $\dot{a} = \dot{\gamma} = 0$ and the frequency–response relation becomes

$$\Omega = 1 + \tfrac{3}{8}\alpha a_0^2 + \tfrac{1}{32}\left(10\beta - \tfrac{15}{8}\alpha^2\right)a_0^4 \pm \tfrac{1}{2}\sqrt{\frac{F^2}{a_0^2} - \mu^2}. \tag{20}$$

Performing the standard stability analysis [14] for the fixed points, one obtains the conditions

$$\begin{aligned} &\mu^2 - 4c > 0 \quad \text{and} \quad c < 0 \quad \text{(unstable solutions)},\\ &\mu^2 - 4c > 0 \quad \text{and} \quad c > 0 \quad \text{(stable solutions)},\\ &\mu^2 - 4c < 0 \quad \text{(stable solutions)}, \end{aligned} \tag{21}$$

where

$$c = \frac{\mu^2}{4} + \frac{F}{2a}\cos\gamma\left\{\tfrac{3}{4}\alpha a^2 + \tfrac{1}{8}\left(10\beta - \tfrac{15}{8}\alpha^2\right)a^4 + \frac{F}{2a}\cos\gamma\right\}. \tag{22}$$

Now, one might wonder if a different choice of ordering would yield better results. Another choice would be to incorporate damping and excitation at the first non-linear order

$$F = \varepsilon f, \quad \mu = \varepsilon\bar{\mu}, \quad \Omega = 1 + \varepsilon\sigma. \tag{23}$$

Note that there is no other choice possible than the considered two different scaling (i.e., Eqs. (23) and (3)). Selecting the excitation and damping to appear at different orders of approximation may lead to inconsistencies. Performing a similar analysis for this choice of ordering yields finally

$$\begin{aligned} \dot{a} = &-\frac{\mu}{2}a + \tfrac{3}{16}\alpha\mu a^3 + \tfrac{1}{8}\mu F\cos\gamma \\ &+ F\sin\gamma\left(\frac{3-\Omega}{4}\right) - \tfrac{9}{32}\alpha F a^2\sin\gamma, \end{aligned} \tag{24}$$

$$\begin{aligned} a\dot{\gamma} = &\, a(\Omega - 1) + \tfrac{1}{8}\mu^2 a - \tfrac{3}{8}\alpha a^3 + F\cos\gamma\left(\frac{3-\Omega}{4}\right) - \tfrac{1}{8}\mu F\sin\gamma \\ &- \frac{1}{32}\left(10\beta - \tfrac{15}{8}\alpha^2\right)a^5 - \tfrac{3}{32}\alpha F a^2\cos\gamma. \end{aligned} \tag{25}$$

Comparing Eqs. (24) and (25) with Eqs. (17) and (18), one readily observes excess terms. The approximate solution is again the one given in Eq. (16) but modulations of amplitudes and phases differ for this choice. More will be said about these different selections in Section 4.

Note that a time transformation $T = \Omega t$ in the beginning of the analysis with the scaling given in Eq. (3) would yield exactly the same frequency–response relation as given in Eq. (20).

3. Approximate solution by MMS II modified

The same problem will be solved by using the modified MMS II due to Lee and Lee [7]. First, the time variable is transformed into

$$T = \Omega t. \tag{26}$$

The equation of motion would then be

$$\Omega^2 x'' + \mu\Omega x' + x + \alpha x^3 + \beta x^5 = F\cos T. \tag{27}$$

Excitation frequency and amplitude and damping are expanded to first non-linear order only [7] and hence

$$\alpha = \varepsilon\bar{\alpha}, \quad \beta = \varepsilon^2\bar{\beta}, \quad F = \varepsilon f, \quad \mu = \varepsilon\bar{\mu},$$
$$\Omega^2 = 1 + \varepsilon\sigma_1, \quad \Omega = 1 + \varepsilon\sigma_2. \tag{28}$$

The approximate expansion is the same as given in Eq. (4) with the time variables defined in a slightly different manner such that $T_n = \varepsilon^n T$. The equations at each order are

$$D_0^2 x_0 + x_0 = 0, \tag{29}$$

$$D_0^2 x_1 + x_1 = -2D_0D_1x_0 - \bar{\mu}D_0x_0 - \sigma_1 D_0^2 x_0 - \bar{\alpha}x_0^3 + f\cos T_0, \tag{30}$$

$$\begin{aligned} D_0^2 x_2 + x_2 = &- 2D_0D_1x_1 - (D_1^2 + 2D_0D_2)x_0 \\ &- \bar{\mu}(D_0x_1 + D_1x_0) - \bar{\mu}\sigma_2 D_0 x_0 \\ &- \sigma_1(D_0^2 x_1 + 2D_0D_1x_0) - 3\bar{\alpha}x_0^2 x_1 - \bar{\beta}x_0^5. \end{aligned} \tag{31}$$

Solutions x_0 and x_1 are the same as given in the previous section. Elimination of secular terms yield the following relations at each order:

$$2\mathrm{i}D_1 A = -\bar{\mu}\mathrm{i}A + \sigma_1 A + \frac{f}{2} - 3\bar{\alpha}A^2\bar{A}, \tag{32}$$

$$2\mathrm{i}D_2 A = -(10\bar{\beta} + \tfrac{3}{8}\bar{\alpha}^2)A^3\bar{A}^2 - \bar{\mu}\sigma_2\mathrm{i}A. \tag{33}$$

The evolution of complex amplitudes in terms of original parameters would be

$$\begin{aligned} 2\mathrm{i}\frac{\mathrm{d}A}{\mathrm{d}T} = &(\Omega^2 - 1)A + \frac{F}{2} - 3\alpha A^2\bar{A} \\ &- (10\beta + \tfrac{3}{8}\alpha^2)A^3\bar{A}^2 - \mu\Omega\mathrm{i}A. \end{aligned} \tag{34}$$

Using the polar form

$$A = \tfrac{1}{2}a\exp(-\mathrm{i}\gamma), \tag{35}$$

the solution- and amplitude-phase relations in real time are

$$x = a\cos(\Omega t - \gamma) + \frac{\alpha a^3}{32}\cos(3\Omega t - 3\gamma) + \cdots, \tag{36}$$

$$\dot{a} = -\frac{\mu}{2}\Omega^2 a + \frac{F}{2}\Omega\sin\gamma, \tag{37}$$

$$a\dot{\gamma} = \frac{\Omega}{2}(\Omega^2 - 1)a - \tfrac{3}{8}\alpha\Omega a^3 - \frac{\Omega}{32}\left(10\beta + \tfrac{3}{8}\alpha^2\right)a^5 + \frac{F}{2}\Omega\cos\gamma. \tag{38}$$

The frequency–response relation is

$$\Omega^2 = 1 + \tfrac{3}{4}\alpha a_0^2 + \tfrac{1}{16}\left(10\beta + \tfrac{3}{8}\alpha^2\right)a_0^4 \pm \sqrt{\frac{F^2}{a_0^2} - (\mu\Omega)^2}. \tag{39}$$

Performing a similar stability analysis yields

$$\begin{aligned} &\mu^2\Omega^4 - 4c > 0 \quad \text{and} \quad c < 0 \quad \text{(unstable solutions)},\\ &\mu^2\Omega^4 - 4c > 0 \quad \text{and} \quad c > 0 \quad \text{(stable solutions)},\\ &\mu^2\Omega^4 - 4c < 0 \quad \text{(stable solutions)}, \end{aligned} \tag{40}$$

where

$$c = \frac{\mu^2\Omega^4}{4} + \frac{F\Omega}{2a}\cos\gamma\left\{\tfrac{3}{4}\alpha\Omega a^2 + \frac{\Omega}{8}\left(10\beta + \tfrac{3}{8}\alpha^2\right)a^4 + \frac{F\Omega}{2a}\cos\gamma\right\}. \tag{41}$$

Next section is devoted to comparisons between the methods as well as numerical solutions

4. Comparisons

The alternative scaling (i.e., Eq. (23)) in MMS I produced very much different amplitude and phase modulation equations (i.e., Eqs. (24) and (25)) compared to the others (i.e., Eqs. (17) and (18) for MMS I and Eqs. (37) and (38) for MMS II modified). This alternative ordering in MMS I produce the extra unphysical solutions which are discussed in detail in Ref. [4]. For this reason, only the solutions corresponding to the best possible choice of MMS I (i.e., orderings given in Eq. (3)) are compared with those of MMS II modified. Comparing the frequency–response relations obtained by each version (i.e., Eq. (20) for MMS I and Eq. (39) for MMS II modified), a major difference is noted in the coefficient of a_0^4. For a suitable selection of positive non-linearity coefficients α and β, the coefficient of a_0^4 can be negative in MMS I, whereas it is always positive in MMS II. The square root terms limit the maximum amplitudes (i.e., root argument is zero) and it is higher in MMS I. MMS I and MMS II frequency–response curves are drawn for the parameter values $\alpha = 0.4$, $\beta = 0.005$, $\mu = 0.02$ and $F = 0.1$ in Fig. 1. Solid lines belong to stable solutions and dotted lines belong to unstable ones. Results of direct numerical integration of the original equation are represented by diamonds in the figure. It is evident that MMS II produces more reliable solutions and as a_0^4 term becomes dominant for larger amplitudes, the error in MMS I solution increases. MMS I predicts up to three stable branches for a given frequency, whereas numerical integration and MMS II both predict up to two stable branches. Time transformation in MMS II is not responsible for the better results. If time transformation had been done for MMS I, exactly the same frequency-response relation given in Eq. (20) would be obtained. For a suitable selection of α and β, it is of course possible to approximately equate the coefficients of a_0^4 for each version in which case both solutions would be indistinguishable.

A sample numerical integration plot is given with the above parameter values for the point corresponding to $\Omega = 1.7$ in Fig. 2. The steady state amplitude is 2.486. In numerical calculations,

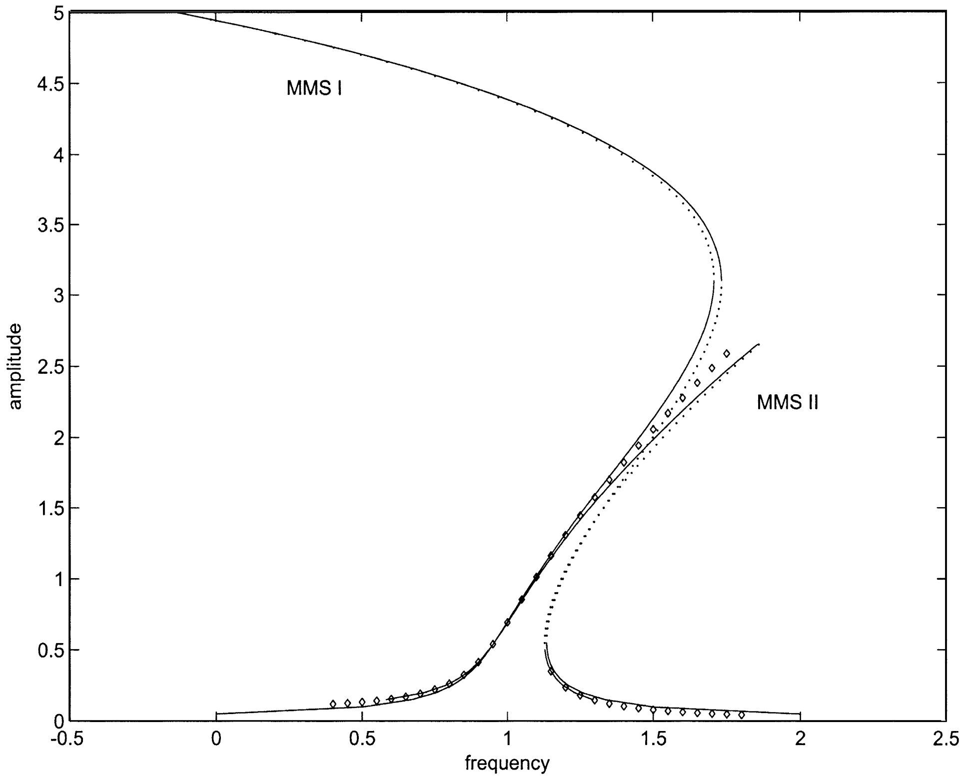

Fig. 1. Frequency–response curves for MMS I and MMS II (solid stable, dotted unstable) and direct numerical results (diamonds) for the non-linear case. ($\alpha = 0.4$, $\beta = 0.005$, $\mu = 0.02$ and $F = 0.1$).

it becomes more and more involved to find specific initial conditions yielding the stable solutions in the neighborhood of the turning points (saddle node bifurcation points). By trying different sets of initial conditions, the highest and lowest values of frequency for jump phenomena occur at $\Omega = 1.75$ and 1.15, respectively. Since extensive trials are needed, these points are not the exact jump points but without appreciable error, it is estimated that jump occurs for slightly higher and lower frequency values.

An interesting special case is linear damped-forced vibrations (i.e., $\alpha = \beta = 0$ in the original equation) where exact closed-form solution is available,

$$x = x_0 \exp\left(-\frac{\mu}{2}t\right) \cos\left(\sqrt{1 - \frac{\mu^2}{4}}\, t + \varphi\right) + \frac{F}{\sqrt{(1 - \Omega^2)^2 + (\mu\Omega)^2}} \cos(\Omega t - \gamma). \tag{42}$$

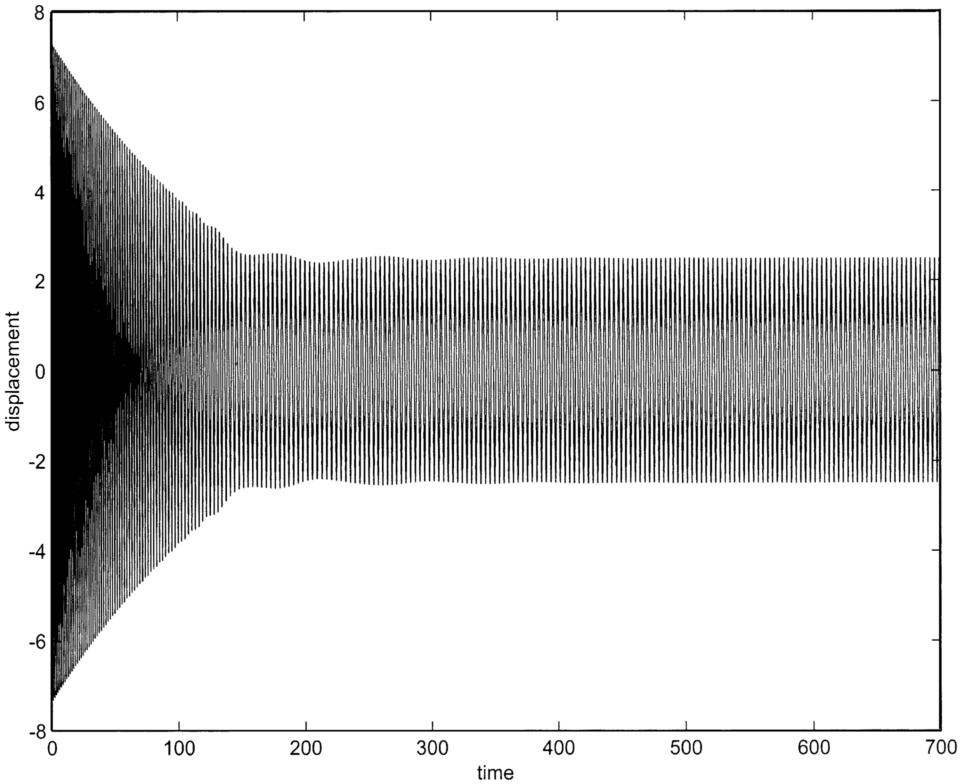

Fig. 2. Time history of the original equation ($x(0) = 0$, $\dot{x}(0) = 30$, $\Omega = 1.7$, $\alpha = 0.4$, $\beta = 0.005$, $\mu = 0.02$ and $F = 0.1$).

The approximate solution using MMS I is

$$x = x_0 \exp\left(-\frac{\mu}{2}t\right)\cos(t+\varphi) + \frac{F}{\sqrt{4(1-\Omega)^2+\mu^2}}\cos(\Omega t - \gamma) \tag{43}$$

and the one using MMS II is

$$x = x_0 \exp\left(-\frac{\mu}{2}\Omega^2 t\right)\cos\left(\Omega\left(1-\frac{\Omega^2-1}{2}\right)t+\varphi\right) + \frac{F}{\sqrt{(1-\Omega^2)^2+(\mu\Omega)^2}}\cos(\Omega t - \gamma). \tag{44}$$

The steady state solution (i.e., the last term) obtained by MMS II is exactly the same with the analytical solution. On the contrary, it is approximately equal to the analytical solution for the case of MMS I and the error increases as the frequency deviates from 1. This can be seen from Fig. 3 clearly where solid lines belong to the analytical as well as MMS II solutions and dashed lines belong to those of MMS I.

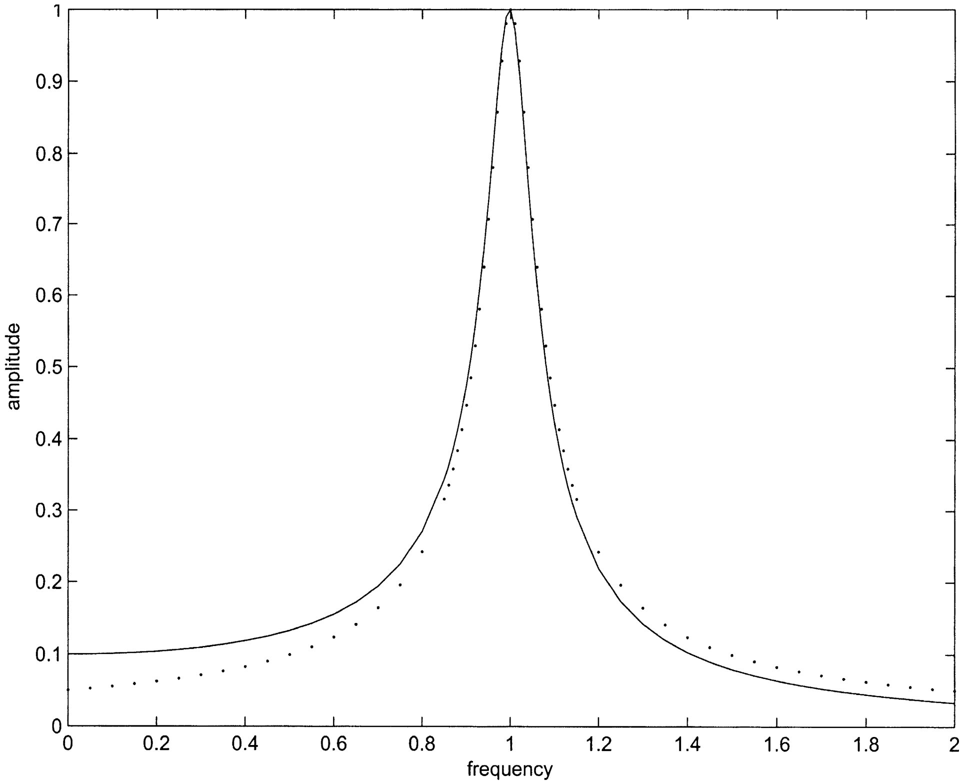

Fig. 3. Frequency–response curves for MMS I (dotted), MMS II (solid) and exact (solid) for the linear case.

Comparing the transient responses (the first terms in the solutions), however, the error introduced is more in MMS II than in MMS I as frequency deviates from 1 as can be readily detected from the arguments of exponential and cosine terms. This conclusion is also supported by the comparisons between the different versions in the general non-linear case. Comparing Eqs. (17) and (18) with Eqs. (37) and (38), one can readily detect the excess multiplications of frequency in the latter case which makes the transient responses imprecise. To see further that MMS I produces more reliable transient solutions, one may consider the more degenerate case of linear damped vibrations without excitation. The approximate analytical solution using MMS I is reproduced from Ref. [3, p. 144]

$$x = x_0 \exp\left(-\frac{\mu}{2}t\right) \cos\left(t - \frac{\mu^2}{2}t + \varphi\right). \tag{45}$$

The solution using MMS II is

$$x = x_0 \exp\left(-\frac{\mu}{2}t\right) \cos(t + \varphi). \tag{46}$$

The damped frequency of the exact analytical solution is $\sqrt{1-\mu^2}$ or approximately $1 - \mu^2/2$. It is clear that the second correction term in the cosine argument can not be retrieved by MMS II. The fail stems from selecting $D_1 A = 0$ at the second order of approximation.

5. Concluding remarks

The following conclusions can be withdrawn from the analysis:

(1) MMS II is better than MMS I in finding steady state solutions.
(2) On the contrary, MMS I is better than MMS II in finding transient responses.

These conclusions are valid for primary resonances of the external excitation. Conclusion 1 confirms the previously obtained results given in Refs. [4,7]. However, in this work, a different non-linear model is used. Conclusion 2 appears to be new and not previously reported. Note that both methods will yield identical results when $D_1 A = 0$ at the first level of approximation [4].

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